TRINITY COLLEGE

Mathematics Methods Unit 3,4
Test 4 2016

Section 1 Calculator Free

Calculus Involving Logarithmic Functions, Continuous Random Variables

STUDENT'S NAME SOLUTIONS

DATE: Friday 22 July **TIME:** 25 minutes **MARKS**: 25

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

Determine the equation of the tangent to the curve $y = x\ln x$ at the point (e,e)

$y' = \ln x + 1$

$x = e$ $\quad m = \ln e + 1$

$= 2$

$y = mx + c$

$y = 2x + c$

(e, e) $\quad e = 2e + c$

$-e = c$

$y = 2x - e$

2. (4 marks)

(a) $\int \frac{\sin x}{1+\cos x} dx$ $= -\int \frac{-\sin x}{1+\cos x} dx$ [2]

$= -\ln|1+\cos x| + c$

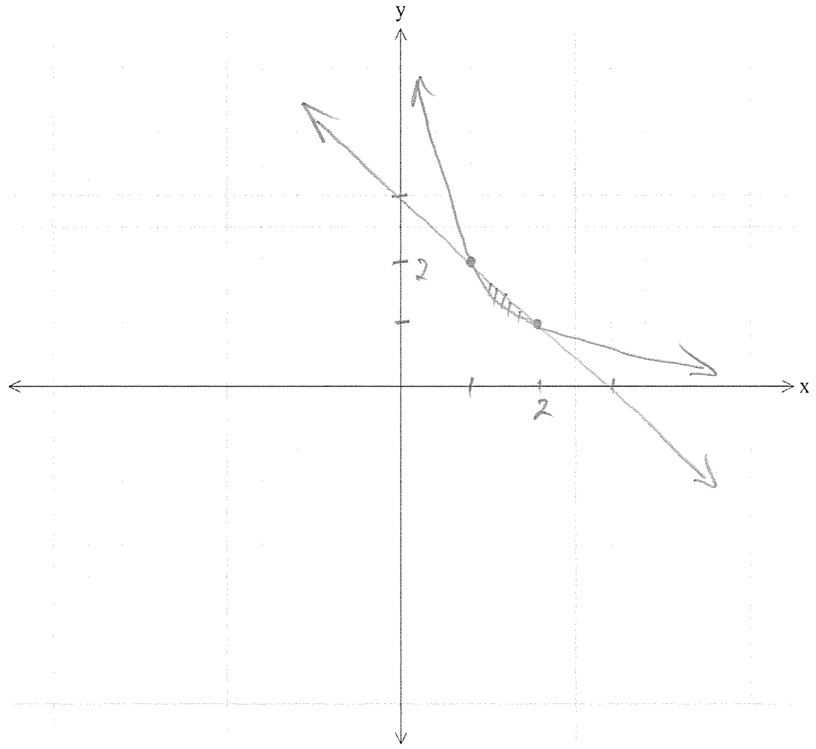
(b) $\int \frac{8-6x^2}{x^3-4x+1} dx$ $= -2\int \frac{3x^2-4}{x^3-4x+1} dx$ [2]

$= -2\ln|x^3-4x+1| + c$

3. (5 marks)

Consider the functions $y = 3 - x$ and $xy = 2$.

(a) Draw a sketch of these two functions which clearly shows the enclosed area. [2]



(b) Determine the exact value of the enclosed area. [3]

$$AREA = \int_1^2 3 - x - \frac{2}{x} \, dx$$

$$= \left[3x - \frac{x^2}{2} - 2\ln x \right]_1^2$$

$$= (6 - 2 - 2\ln 2) - (3 - \tfrac{1}{2} - 2\ln 1)$$

$$= \frac{3}{2} - \ln 4$$

4. (12 marks)

Differentiate each of the following functions. Do NOT simplify.

(a) $y = \ln \frac{2x}{x^2 - 1}$ $= \ln 2x - \ln (x^2 - 1)$ [3]

$$y' = \frac{2}{2x} - \frac{2x}{x^2 - 1}$$

(b) $y = \ln \tan 2x$ $= \ln \frac{\sin 2x}{\cos 2x}$ [3]

$$= \ln \sin 2x - \ln \cos 2x$$

$$y' = \frac{2\cos 2x}{\sin 2x} - \left(\frac{-2\sin 2x}{\cos 2x} \right)$$

(c) $y = \ln \ln x^2$ $y' = \dfrac{\frac{2x}{x^2}}{\ln x^2}$ [3]

(d) $y = \ln(e^x(1 - e^{-x}))$ $= \ln e^x + \ln (1 - e^{-x})$ [3]

$$= x + \ln (1 - e^{-x})$$

$$y' = 1 + \frac{e^{-x}}{1 - e^{-x}}$$

TRINITY COLLEGE

Mathematics Methods Unit 3,4
Test 4 2016

Section 2 Calculator Assumed

Calculus Involving Logarithmic Functions, Continuous Random Variables

STUDENT'S NAME ______________________________

DATE: Friday 22 July **TIME:** 30 minutes **MARKS:** 29

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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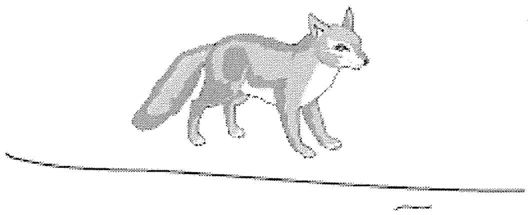
5. (5 marks)

The time, t, in hours that a fox spends hunting each night is a continuous random variable with probability density function

$$f(t)=\begin{cases}\dfrac{k}{32}t(4-t) & \text{for } 0\le t\le 4\\ 0 & \text{otherwise}\end{cases}$$



(a) Determine the value of k. [3]

$$\frac{k}{32}\int_0^4 (4t-t^2)\,dt = 1$$

$$\left[2t^2-\frac{t^3}{3}\right]_0^4 = \frac{32}{k}$$

$$32-\frac{64}{3} = \frac{32}{k}$$

$$\frac{32}{3} = \frac{32}{k}$$

$$k = 3$$

(b) Calculate the probability the fox spends more than 3 hours hunting on one night. [2]

$$\int_3^4 \frac{3}{32}(4t-t^2)\,dt = 0.1563$$

6. (18 marks)

The time, in minutes, between telephone calls received at a pizza shop is a continuous random variable, T , with a density function given by

$$f(t)=\begin{cases}0.25e^{-0.25t} & for\ t\geq 0\\ 0 & elsewhere\end{cases}$$

(a) Calculate the probability that the next call occurs within 8 minutes. [2]

$\int_0^8 0.25e^{-0.25t}\,dt = 0.8647$

(b) Calculate the probability that the next call occurs between 3 and 6 minutes given it occurs within 8 minutes. [2]

$P(3 \leq x \leq 6 / x \leq 8) = \dfrac{0.2492}{0.8647}$

$= 0.2882$

(c) Determine the expected time to the next call. [3]

$E(x) = \int_0^\infty x \times 0.25e^{-0.25x}\,dx$

$= 4$

(d) Determine the interval of time that is within one standard deviation of the expected completion time. [4]

$VAR = \int_0^\infty (x-4)^2\, 0.25e^{-0.25x}\,dx$

$= 16$

$SD = 4$

INTERVAL $0 \leq t \leq 8$

(e) Determine $\operatorname{Var}(1-2T)$, where Var is the variance. [2]

$SD = 4$

$|-2 \times SD| = 8$

$VAR = 8^2$

$= 64$

(f) (i) For the random variable T, give the cumulative distribution function $F(t)$. [3]

$F(t) = \int_0^t 0.25 e^{-0.25x} dx$

$= 1 - e^{-0.25t}$

$$P(T \leq t) = \begin{cases} 0 & t \leq 0 \\ 1 - e^{-0.25t} & t > 0 \end{cases}$$

(ii) Determine $P(T \geq 10)$ [2]

$\int_{10}^{\infty} 0.25 e^{-0.25x} dx = 0.082$

OR

$1 - F(10) = 1 - 0.9179$

$= 0.0821$

7. (6 marks)

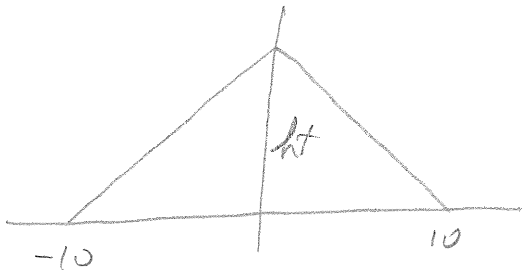
X is a continuous random variable, denoting the number of minutes in excess of two hours which a person takes to travel from one town to another. The probability density function is defined as follows.

$$f(x)=\begin{cases} k(10+x) & -10 \le x < 0 \\ k(10-x) & 0 \le x \le 10 \\ 0 & elsewhere \end{cases}$$

(a) Determine the value of k. [4]

$\frac{1}{2} \times 20 \times ht = 1$

$ht = \frac{1}{10}$



$y = k(10+x)$

$(0, \frac{1}{10})$ $\quad \frac{1}{10} = 10k$

$\frac{1}{100} = k$

(b) Determine the probability that a person will take longer than 115 minutes to reach the next town. [2]

$P(X \geq -5) = 0.875$